

A note on Connections and Bimodules

Aristophanes Dimakis

Institut für Theoretische Physik
Bunsenstr. 9, D-37073 Göttingen, Germany

Abstract

We discuss three problems related to connections on bimodules. These are left and right Leibniz rule for connections, left and right linearity of their curvatures and extension of connections to tensor products of modules.

1. Recently in a series of papers [1], [2] and [3] the concept of a connection on a bimodule was studied. If \mathcal{A} is an algebra and $(\Omega(\mathcal{A}), d)$ a differential calculus over \mathcal{A} , and M an \mathcal{A} -bimodule, a right-module connection on M is a map

$$\nabla : M \rightarrow M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) , \quad (1)$$

satisfying for $f \in \mathcal{A}$ and $a \in M$ the *right Leibniz rule*

$$\nabla(af) = (\nabla a)f + a \otimes_{\mathcal{A}} df . \quad (2)$$

In order to take account of the left action of \mathcal{A} on M the authors of the above papers use a generalized permutation

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} M \rightarrow M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) , \quad (3)$$

and impose additionally a *left Leibniz rule*

$$\nabla(fa) = \sigma(df \otimes_{\mathcal{A}} a) + f \nabla a . \quad (4)$$

It turns out that this implies a strong restriction on ∇ , a fact which reveals a close correlation between the differential calculus over \mathcal{A} and the geometry on M .

Evidence for (4) appeared also in [4]. There it was shown that under general conditions, if N is a right \mathcal{A} -module with a connection $\nabla' : N \rightarrow N \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, then a necessary and sufficient condition for ∇ and ∇' to define a connection $\nabla_{\otimes} : N \otimes_{\mathcal{A}} M \rightarrow N \otimes_{\mathcal{A}} M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$

in a fashion similar to the commutative case is that ∇ satisfies the left Leibniz rule (4) and

$$\nabla_{\otimes}(b \otimes_{\mathcal{A}} a) := (\text{id}_N \otimes_{\mathcal{A}} \sigma)(\nabla' b) \otimes_{\mathcal{A}} a + b \otimes_{\mathcal{A}} \nabla a . \quad (5)$$

A further point related to the left Leibniz rule is the tensor character of the curvature of the connection. Extending ∇ to

$$\nabla : M \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \rightarrow M \otimes_{\mathcal{A}} \Omega(\mathcal{A}) , \quad (6)$$

by

$$\nabla(a \otimes_{\mathcal{A}} \omega) = (\nabla a)\omega + a \otimes_{\mathcal{A}} d\omega , \quad (7)$$

the curvature of ∇ is the map $\nabla^2 : M \otimes_{\mathcal{A}} \Omega \rightarrow M \otimes_{\mathcal{A}} \Omega$. It is easy to see that although this is a right Ω -homomorphism it is not even a left \mathcal{A} -homomorphism. In fact if one tries to calculate left linearity, one finds that the action of ∇ on σ must be specified. There is no evidence however how this has to be done. This is the reason why in [5] it is proposed the factoring out of the submodule of $M \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$ generated by $\nabla^2(fa) - f\nabla^2 a$, for all $f \in \mathcal{A}$ and $a \in M$. On the quotient space the curvature is both left and right \mathcal{A} -linear.

In the sequel we try to shed some new light on these constructions using the fact that for an \mathcal{A} -bimodule M the left action of \mathcal{A} on M induces a canonical imbedding of \mathcal{A} into the algebra of right \mathcal{A} -endomorphisms of M .

2. Let $\text{End}^{\mathcal{A}}(M)$ denote the ring of right \mathcal{A} -endomorphism of M , i.e. for $\phi \in \text{End}^{\mathcal{A}}(M)$, $a \in M$ and $f \in \mathcal{A}$ we have $\phi(af) = \phi(a)f$. A right module connection on M induces a map

$$\hat{\nabla} : \text{End}^{\mathcal{A}}(M) \rightarrow \text{Hom}^{\mathcal{A}}(M, M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})) , \quad (8)$$

defined by

$$\hat{\nabla}(\phi) := \nabla \circ \phi - \phi \circ \nabla , \quad (9)$$

where ϕ in the last term above denotes also the obvious *extension* of $\phi : M \rightarrow M$ to $\phi : M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ given by $\phi(a \otimes_{\mathcal{A}} \alpha) := (\phi a) \otimes_{\mathcal{A}} \alpha$. $\text{Hom}^{\mathcal{A}}(M, M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}))$ denotes the additive group of right \mathcal{A} -homomorphisms. With the aid of the extension mentioned, it becomes in a natural way a $\text{End}^{\mathcal{A}}(M)$ -bimodule.

Since M is a \mathcal{A} -bimodule there is a canonical algebra homomorphism $\kappa_0 : \mathcal{A} \rightarrow \text{End}^{\mathcal{A}}(M)$ with

$$f \mapsto \hat{f} := \kappa_0(f) , \quad \hat{f}(a) := fa . \quad (10)$$

The action of $\hat{\nabla}$ on \hat{f} is given by

$$(\hat{\nabla} \hat{f})a = \nabla(fa) - f\nabla a . \quad (11)$$

Let $\Omega_{\nabla}^1(\mathcal{A})$ be the \mathcal{A} -bimodule, which is the additive subgroup of $\text{Hom}^{\mathcal{A}}(M, M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}))$, generated by $(\kappa_0 \mathcal{A})(\hat{\nabla} \kappa_0 \mathcal{A})(\kappa_0 \mathcal{A})$, and where for $f, g \in \mathcal{A}$ and $\Phi \in \Omega_{\nabla}^1(\mathcal{A})$ we set

$$f \Phi g := \hat{f} \circ \Phi \circ \hat{g} . \quad (12)$$

Let $d_{\nabla} := \hat{\nabla} \circ \kappa_0$, then it is clear that

$$d_{\nabla} : \mathcal{A} \rightarrow \Omega_{\nabla}^1(\mathcal{A}) , \quad (13)$$

is a derivation since

$$\hat{\nabla}(\hat{f} \circ \hat{g}) = (\hat{\nabla} \hat{f}) \circ \hat{g} + \hat{f} \circ (\hat{\nabla} \hat{g}) . \quad (14)$$

Hence $(\Omega_{\nabla}^1(\mathcal{A}), d_{\nabla})$ defines a first order differential calculus over \mathcal{A} ,

$$\nabla(fa) = (d_{\nabla} f)a + f \nabla a , \quad (15)$$

and the left Leibniz rule uses this new differential calculus instead of $\Omega(\mathcal{A})$.

If $(\Omega_u^1(\mathcal{A}), d_u)$ denotes the universal first order differential calculus over \mathcal{A} , then there is a homomorphism $\kappa_1 : \Omega_u^1(\mathcal{A}) \rightarrow \Omega_{\nabla}^1(\mathcal{A}_M)$ such that $\kappa_1 \circ d_u = d_{\nabla}$, i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d_u} & \Omega_u^1(\mathcal{A}) \\ \parallel & & \kappa_1 \downarrow \\ \mathcal{A} & \xrightarrow{d_{\nabla}} & \Omega_{\nabla}^1(\mathcal{A}) \end{array} \quad (16)$$

Define now $\sigma_u : \Omega_u^1(\mathcal{A}) \otimes_{\mathcal{A}} M \rightarrow M \otimes_{\mathcal{A}} \Omega_u^1(\mathcal{A})$ by

$$\sigma_u(\alpha \otimes_{\mathcal{A}} a) := \kappa_1(\alpha)(a) , \quad (17)$$

for $\alpha \in \Omega_u^1(\mathcal{A})$ and $a \in M$, then (15) takes the form

$$\nabla(fa) = \sigma_u(d_u f \otimes_{\mathcal{A}} a) + f \nabla a . \quad (18)$$

The existence of σ_u was first proved in [2]. For σ as in (4) to exist, κ_1 must factor uniquely through the projection $\pi_1 : \Omega_u^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A})$, that is $\kappa_1 = \hat{\kappa}_1 \circ \pi_1$ with a homomorphism $\hat{\kappa}_1 : \Omega^1(\mathcal{A}) \rightarrow \Omega_{\nabla}^1(\mathcal{A})$ and $d_{\nabla} = \hat{\kappa}_1 \circ d$. Then

$$\sigma(\alpha \otimes_{\mathcal{A}} a) := \hat{\kappa}_1(\alpha)(a) , \quad (19)$$

for $\alpha \in \Omega^1(\mathcal{A})$ and $a \in M$.

We conclude that a connection ∇ on a bimodule M satisfies a left Leibniz rule (4), if $(\Omega_{\nabla}^1(\mathcal{A}), d_{\nabla}) \preceq (\Omega^1(\mathcal{A}), d)$. Here \preceq is the obvious partial ordering on the set of classes of isomorphic first order differential calculi over \mathcal{A} , where $(\Omega_1^1(\mathcal{A}), d_1) \preceq (\Omega_2^1(\mathcal{A}), d_2)$ if a homomorphism $\rho : \Omega_2^1(\mathcal{A}) \rightarrow \Omega_1^1(\mathcal{A})$ exists, such that $d_1 = \rho \circ d_2$.

3. As $\nabla : M \rightarrow M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ extends to $\nabla : M \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \rightarrow M \otimes_{\mathcal{A}} \Omega(\mathcal{A})$ so also $\hat{\nabla} : \text{End}^{\mathcal{A}}(M) \rightarrow \text{Hom}^{\mathcal{A}}(M, M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}))$ extends to

$$\hat{\nabla} : \text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A})) \rightarrow \text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A})) , \quad (20)$$

where $\text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ denotes the ring of right $\Omega(\mathcal{A})$ -homomorphisms of $M \otimes_{\mathcal{A}} \Omega(\mathcal{A})$. Denote by $\text{End}_r^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ the additive group of endomorphisms¹ of degree r , then $\text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ is a graded ring and $\text{End}_0^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A})) \cong \text{End}^{\mathcal{A}}(M)$ by the extension of $\phi \in \text{End}^{\mathcal{A}}(M)$ to $\phi \in \text{End}_0^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ with

$$\phi(a \otimes_{\mathcal{A}} \omega) := \phi(a) \otimes_{\mathcal{A}} \omega . \quad (21)$$

For $\Phi \in \text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ of degree r we set

$$\hat{\nabla}\Phi := \nabla \circ \Phi - (-1)^r \Phi \circ \nabla . \quad (22)$$

Now let $\hat{\Omega}$ denote the graded subalgebra of $\text{End}^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ generated by $\kappa_0 \mathcal{A}$ and $\hat{\nabla}$, i.e. the elements of $\hat{\Omega}$ are finite linear combinations of expressions of the form

$$(\hat{\nabla}^{k_1} \hat{f}_1) \circ \dots \circ (\hat{\nabla}^{k_r} \hat{f}_r) ,$$

with $\hat{\nabla}^0 \hat{f} := \hat{f}$. Obviously $\hat{\nabla}$ restricts to $\hat{\nabla} : \hat{\Omega} \rightarrow \hat{\Omega}$. Note that although $\hat{\nabla}$ is a graded derivation, i.e. for $\Phi \in \hat{\Omega}^r$ and $\Psi \in \hat{\Omega}$ we have

$$\hat{\nabla}(\Phi \circ \Psi) = (\hat{\nabla}\Phi) \circ \Psi + (-1)^r \Phi \circ (\hat{\nabla}\Psi) , \quad (23)$$

the pair $(\hat{\Omega}, \hat{\nabla})$ fails to be a differential algebra over $\kappa_0 \mathcal{A}$ since

$$(\hat{\nabla}^2 \hat{f})(a) = \nabla^2(fa) - f\nabla^2(a) , \quad (24)$$

does not vanish in general.

Set $J = \bigoplus_r J^r$ for the graded right $\Omega(\mathcal{A})$ -submodule of $M \otimes_{\mathcal{A}} \Omega(\mathcal{A})$ generated by elements of the form

$$(\hat{\nabla}^2 \Phi)\xi \quad \text{for all} \quad \xi \in M \otimes_{\mathcal{A}} \Omega(\mathcal{A}) , \quad \Phi \in \hat{\Omega} .$$

It is important to note that $\nabla J \subset J$ and $\Phi J \subset J$ for all $\Phi \in \hat{\Omega}$. Hence taking the factor module $\Omega(M) := [M \otimes_{\mathcal{A}} \Omega(\mathcal{A})]/J$ with canonical projection p , the elements $\Phi \in \hat{\Omega}$ have unique factorizations $p \circ \Phi = \hat{\Phi} \circ p$ with

$$\hat{\Phi} : \Omega(M) \rightarrow \Omega(M) .$$

The same holds for ∇ , although we do not introduce a new symbol for $\nabla : \Omega(M) \rightarrow \Omega(M)$. Note that $\Omega(M)$ is a left \mathcal{A} -right $\Omega(\mathcal{A})$ -bimodule.

¹That is for $\Phi \in \text{End}_r^{\Omega}(M \otimes_{\mathcal{A}} \Omega(\mathcal{A}))$ and for all $\xi \in M \otimes_{\mathcal{A}} \Omega^s(\mathcal{A})$ with arbitrary s we have $\Phi(\xi) \in M \otimes_{\mathcal{A}} \Omega^{r+s}(\mathcal{A})$.

Let $\Omega_{\nabla}(\mathcal{A})$ be the graded algebra of the factors $\hat{\Phi}$ for all $\Phi \in \hat{\Omega}$. Note that since $J^0 = J^1 = \{0\}$, we put $\Omega_{\nabla}^0(\mathcal{A}) = \mathcal{A}$; $\Omega_{\nabla}^1(\mathcal{A})$ is the \mathcal{A} -bimodule defined in (12). We denote with $\hat{p} : \hat{\Omega} \rightarrow \Omega_{\nabla}(\mathcal{A})$ the algebra homomorphism which sends Φ to $\hat{\Phi}$. It is easy to see now that $\hat{\nabla}$ factors through \hat{p} . We set $\hat{p} \circ \hat{\nabla} = d_{\nabla} \circ \hat{p}$ and find that d_{∇} is a graded derivation of $\hat{\Omega}(\mathcal{A}_M)$ which satisfies additionally $d_{\nabla}^2 = 0$. Hence the pair $(\Omega_{\nabla}(\mathcal{A}), d_{\nabla})$ defines a differential calculus over \mathcal{A} and the diagram in (16) extends to

$$\begin{array}{ccc} \Omega_u(\mathcal{A}) & \xrightarrow{d_u} & \Omega_u(\mathcal{A}) \\ \kappa \downarrow & & \kappa \downarrow \\ \Omega_{\nabla}(\mathcal{A}) & \xrightarrow{d_{\nabla}} & \Omega_{\nabla}(\mathcal{A}) \end{array} \quad (25)$$

Now let

$$\sigma_u : \Omega_u(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(M) \rightarrow \Omega(M) ,$$

be the right $\Omega(\mathcal{A})$ -homomorphism defined for $\omega \in \Omega_u(\mathcal{A})$, $\xi \in \Omega(M)$ by

$$\sigma_u(\omega \otimes_{\mathcal{A}} \xi) := \kappa(\omega)(\xi) . \quad (26)$$

Then σ_u is graded in the first factor, right $\Omega(\mathcal{A})$ -linear in the second factor with $\sigma_u(f \otimes_{\mathcal{A}} \xi) = f\xi$, and satisfies for $\omega_1, \omega_2 \in \Omega_u(\mathcal{A})$ and $\omega \in \Omega_u^r(\mathcal{A})$

$$\sigma_u(\omega_1 \omega_2 \otimes_{\mathcal{A}} \xi) = \sigma_u(\omega_1 \otimes_{\mathcal{A}} \sigma_u(\omega_2 \otimes_{\mathcal{A}} \xi)) \quad (27)$$

$$\nabla \sigma_u(\omega \otimes_{\mathcal{A}} \xi) = \sigma_u(d_u \omega \otimes_{\mathcal{A}} \xi) + (-1)^r \sigma_u(\omega \otimes_{\mathcal{A}} \nabla \xi) . \quad (28)$$

Again if $\kappa = \hat{\kappa} \circ \pi$ holds with $\hat{\kappa} : \Omega(\mathcal{A}) \rightarrow \Omega_{\nabla}(\mathcal{A})$, then for $\omega \in \Omega(\mathcal{A})$

$$\sigma(\omega \otimes_{\mathcal{A}} \xi) := \hat{\kappa}(\omega)(\xi) , \quad (29)$$

defines a map $\sigma : \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(M) \rightarrow \Omega(M)$ with similar properties as σ_u .

Here again we find that a σ exists if, and only if we have $(\Omega_{\nabla}(\mathcal{A}), d_{\nabla}) \preceq (\Omega(\mathcal{A}), d)$, where now \preceq is a partial ordering on the set of all classes of isomorphic differential calculi over \mathcal{A} , defined in a similar way as \preceq in the last paragraph for first order differential calculi. In fact this set has the structure of a lattice, induced on it by the lattice of differential ideals of $(\Omega_u(\mathcal{A}), d_u)$.

4. Let N be a right \mathcal{A} -module and let

$$\nabla'_M : N \rightarrow N \otimes_{\mathcal{A}} \Omega_{\nabla}^1(\mathcal{A}) ,$$

be a connection on N with respect to the differential calculus over \mathcal{A} defined by $\nabla : M \rightarrow M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. Let $N_0 := \{b \in N \mid b \otimes_{\mathcal{A}} a = 0 \text{ for all } a \in M\}$ and $M_0 := \{a \in M \mid b \otimes_{\mathcal{A}} a = 0 \text{ for all } b \in N\}$, in the following we assume that $\nabla M_0 \subset M_0 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ and $\nabla'_M N_0 \subset N_0 \otimes_{\mathcal{A}} \Omega_{\nabla}^1(\mathcal{A})$.

A connection on the tensor product $N \otimes_{\mathcal{A}} M$

$$\nabla_{\otimes} : N \otimes_{\mathcal{A}} M \rightarrow N \otimes_{\mathcal{A}} M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) ,$$

is defined by

$$\nabla_{\otimes}(b \otimes_{\mathcal{A}} a) := (\nabla'_M b)a + b \otimes_{\mathcal{A}} \nabla a , \quad (30)$$

where the first term on the right side has to be interpreted in the sense of $(b \otimes_{\mathcal{A}} \hat{\Phi})a := b \otimes_{\mathcal{A}} (\hat{\Phi}a)$, for $\hat{\Phi} \in \Omega^1_{\nabla}(\mathcal{A})$.

As explained before, in case $(\Omega_{\nabla}(\mathcal{A}), d_{\nabla}) \preceq (\Omega(\mathcal{A}), d)$ we have a homomorphism $\hat{\kappa} : \Omega(\mathcal{A}) \rightarrow \Omega_{\nabla}(\mathcal{A})$ with $d_{\nabla} = \hat{\kappa} \circ d$, then we have also a homomorphism $\hat{\nu} : N \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \rightarrow N \otimes_{\mathcal{A}} \Omega_{\nabla}(\mathcal{A})$ with

$$\hat{\nu}(b \otimes_{\mathcal{A}} \omega) = b \otimes_{\mathcal{A}} \hat{\kappa}(\omega) , \quad (31)$$

for $\omega \in \Omega(\mathcal{A})$.

If $\nabla' : N \rightarrow N \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is a connection on N , with respect to the original differential calculus over \mathcal{A} then one can define a connection $\nabla_{\otimes} : N \otimes_{\mathcal{A}} M \rightarrow N \otimes_{\mathcal{A}} M \otimes_{\mathcal{A}} \Omega(\mathcal{A})$ by setting

$$\nabla_{\otimes}(b \otimes_{\mathcal{A}} a) := \hat{\nu}(\nabla' b)a + b \otimes_{\mathcal{A}} \nabla a , \quad (32)$$

if $\nabla' N_0 \subset N_0 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. In this case ∇' defines a unique associated connection ∇'_M making the diagram

$$\begin{array}{ccc} N \otimes_{\mathcal{A}} \Omega(\mathcal{A}) & \xrightarrow{\nabla'} & N \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \\ \hat{\nu} \downarrow & & \hat{\nu} \downarrow \\ N \otimes_{\mathcal{A}} \Omega_{\nabla}(\mathcal{A}) & \xrightarrow{\nabla'_M} & N \otimes_{\mathcal{A}} \Omega_{\nabla}(\mathcal{A}) \end{array} \quad (33)$$

commute.

5. We found that a right-module connection ∇ on a bimodule M over an algebra \mathcal{A} , with differential calculus $(\Omega(\mathcal{A}), d)$ defines a new differential calculus $(\Omega_{\nabla}(\mathcal{A}), d_{\nabla})$ over \mathcal{A} , in general different from the original calculus. With respect to this calculus ∇ satisfies a left Leibniz rule. If there is an algebra homomorphism $\rho : \Omega(\mathcal{A}) \rightarrow \Omega_{\nabla}(\mathcal{A})$, such that $d_{\nabla} \circ \rho = \rho \circ d$ then we can use ρ to interpret the left Leibniz rule to be with respect to $(\Omega(\mathcal{A}), d)$. Otherwise we must live with two or more different differential calculi on \mathcal{A} simultaneously. In any case it is important to study, how far the induced differential calculus $(\Omega_{\nabla}(\mathcal{A}), d_{\nabla})$ can be from the original one $(\Omega(\mathcal{A}), d)$ for given bimodule M .

Acknowledgment. I would like to thank F. Müller-Hoissen for stimulating discussions and critical comments.

References

- [1] Mourad J 1995, *Linear Connections in Non-Commutative Geometry*, Class. Quant. Grav. **12** 965; Dubois-Violette M, Michor P 1995, *Connections on Central Bimodules*, Preprint LPTHE Orsay 94/100.
- [2] Dubois-Violette M and Masson T 1995, *On the First-Order Operators in Bimodules*, Preprint LPTHE Orsay 95/56.
- [3] Dubois-Violette M, Madore J, Masson T and Mourad J 1995, *Linear Connections on the Quantum Plane*, Lett. Math. Phys. **35** 351; Madore J, Masson T and Mourad J 1995, *Linear Connections on Matrix Geometries*, Class. Quant. Grav. **12** 1429.
- [4] Bresser K, Müller-Hoissen F, Dimakis A and Sitarz A 1995, *Noncommutative Geometry of Finite Groups*, Preprint GOET-TP 95/95, q-alg/9509004.
- [5] Dubois-Violette M, Madore J, Masson T and Mourad J 1995, *On Curvature in Non-commutative Geometry*, Preprint LPTHE Orsay 95/63, q-alg/9512004.